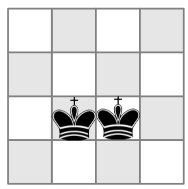
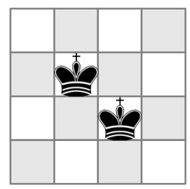
CS103 Handout 27
Spring 2017 May 9, 2017

Extra Practice Problems 5

A Clash of Kings

Chess is a game played on an 8×8 grid with a variety of pieces. In chess, no two king pieces can ever occupy two squares that are immediately adjacent to one another horizontally, vertically, or diagonally. For example, the following positions are illegal:

Prove that it is impossible to legally place 17 kings onto a chessboard.

Induction and Strict Orders

Let A be a set and $<_A$ be a strict order over A. Recall from Problem Set Four that a ***chain in*** $<_A$ is a series of elements $x_1, \ldots, x_n$ drawn from A such that

$$x_1 <_A x_2 <_A \ldots <_A x_n.$$

Prove, by induction, that if $x_1, \ldots, x_n$ is a chain in $<_A$ with $n \geq 2$ elements, then $x_1 <_A x_n$.

Strengthening Relations

Let's introduce a new definition. Let R and T be binary relations over the same set A. We'll say that R is ***no stronger than*** than T if the following statement is true:

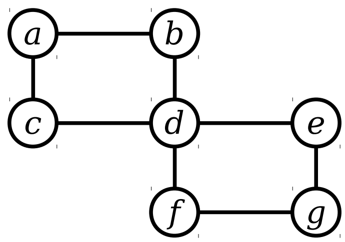
$$\forall a \in A. \forall b \in A. (aRb \to aTb)$$

i. Let R and T be binary relations over the same set A where R is no stronger than T. Prove or disprove: if R is an equivalence relation, then T is an equivalence relation.

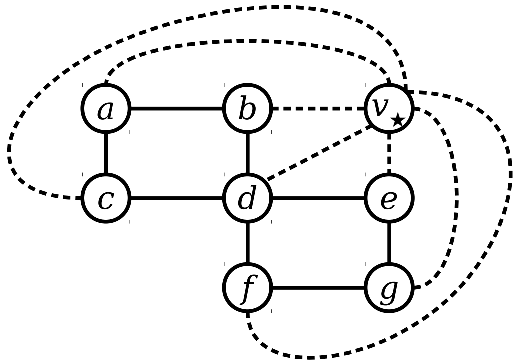
ii. Let R and T be binary relations over the same set A where R is no stronger than T. Prove or disprove: if T is an equivalence relation, then R is an equivalence relation.

Outerplanar Graphs

If G is a graph, the ***augmentation*** of G, denoted Aug(G), is formed by adding a new node $v_\star$ to G, then adding edges from $v_\star$ to each other node in G. For example, below is a graph G and its augmentation Aug(G). To make it easier to see the changes between G and Aug(G), we've drawn the edges added in Aug(G) using dashed lines:



Graph G

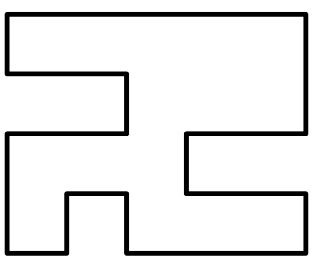


Aug(G)

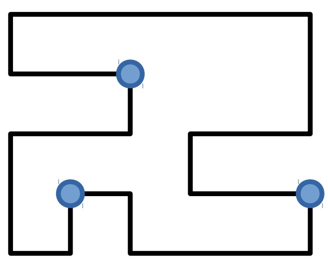
Here's one additional definition: an undirected graph G is called an ***outerplanar graph*** if Aug(G) is a planar graph. In other words, if Aug(G) is a *planar* graph, then the original graph G is an *outerplanar* graph.

i. Using the four-color theorem about planar graphs, prove the ***three-color theorem***: every outerplanar graph is 3-colorable.

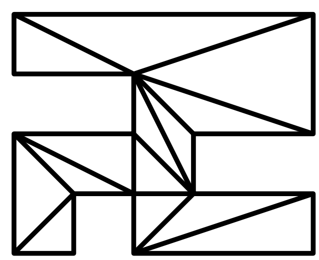
Here's a nifty application of outerplanar graphs. Imagine that you have a room in the shape of a polygon. You're interested in placing floodlights in some number of the corners of the room so that the entire room will be illuminated. You can always illuminate the entire room by putting floodlights in all the corners of the room, and the challenge is to find a way to minimize the number of necessary lights. For example, here's one possible room and one set of three floodlights that would illuminate the room:



The Room



Three Floodlights



A Triangulation

A useful concept for modeling this problem is ***polygon triangulation***. Given a polygon, a triangulation of that polygon is a way of adding extra internal lines connecting the existing vertices of that polygon so that (1) the polygon ends up subdivided into non-overlapping triangles and (2) no new vertices are added. One possible triangulation of the original room is shown above. Importantly, *any floodlight placed at the corner of a triangle will illuminate everything in that triangle*, since there's nothing to obstruct the light.

You can think about the triangulation of a polygon as a planar graph: each vertex is a node, and each line is an edge. But more than that, the triangulation of any polygon is an *outerplanar* graph, since the augmentation is always planar. (You don't need to prove this)

ii. Using your result from part (i) and the fact that any polygon can be triangulated, prove that you can always illuminate a room in the shape of an n-vertex polygon with at most $\lfloor n/3 \rfloor$ floodlights. *(Hint: If you have a 3-coloring of a triangulated polygon, what must be true about any triangle's corners?)*

Odd and Even Functions

Up to this point, most of our discussion of functions has involved functions from arbitrary domains to arbitrary codomains. If we restrict ourselves to functions with specific types of domains and codomains, then we can start exploring more nuanced and interesting classes of functions.

Let's suppose that we have a function $f : \mathbb{R} \to \mathbb{R}$. We'll say that f is an ***odd function*** if the following is true:

$$\forall x \in \mathbb{R}.\ f(-x) = -f(x)$$

This function explores properties of odd functions.

i. Prove that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are odd, then $g \circ f$ is also odd.

ii. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is odd and is a bijection, then f^{-1} is also odd.

We can define ***even functions*** as follows. A function $f : \mathbb{R} \to \mathbb{R}$ is called ***even*** if the following is true:

$$\forall x \in \mathbb{R}.\ f(-x) = f(x)$$

iii. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an even function, then f is ***not*** a bijection.

It turns out that every function $f : \mathbb{R} \to \mathbb{R}$ can be written as the sum of an odd function and an even function. The next few parts of this problem ask you to prove this.

iv. Let $f : \mathbb{R} \to \mathbb{R}$ be an odd function. Prove that for any $r \in \mathbb{R}$, the function $r \cdot f : \mathbb{R} \to \mathbb{R}$ defined as $(r \cdot f)(x) = r \cdot f(x)$ is also odd.

v. Let $f : \mathbb{R} \to \mathbb{R}$ be an even function. Prove that for any $r \in \mathbb{R}$, the function $r \cdot f : \mathbb{R} \to \mathbb{R}$ defined as $(r \cdot f)(x) = r \cdot f(x)$ is also even.

vi. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Prove that $g : \mathbb{R} \to \mathbb{R}$ defined as $g(x) = f(x) - f(-x)$ is odd.

vii. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Prove that $h : \mathbb{R} \to \mathbb{R}$ defined as $h(x) = f(x) + f(-x)$ is even.

viii. Prove that any function $f : \mathbb{R} \to \mathbb{R}$ can be expressed as the sum of an odd function and an even function.